

Wess–Zumino terms and Duality

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Abstract

We show the equivalence between Stückelberg and Wess–Zumino methods of restoration of gauge symmetries of the anomalous, Abelian, effective action, in arbitrary even dimensions $D = 2k$. We present dual version of Wess–Zumino terms with the compensating field described by a Kalb Ramond like $p = 2k - 2$ form.

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By now we know that the anomalous behavior of classical symmetries caused by quantum matter effects is encoded in the anomalous effective action either of gauge fields or gravity [1], [2]. Anomalous effective action reflects the incompatibility of various classical symmetries at the quantum level. Therefore, the best one can do quantum mechanically is to preserve one of the symmetry on the expense of the other in non-chiral models, while in chiral models even this choice is not possible. Various proposals to restore broken symmetries at the quantum level have been put forward. One of these proposals, largely exploited in the literature, is based on *ad hoc* construction of additional *local* terms in the anomalous effective Lagrangian, known as Wess–Zumino terms [3], which restore the broken symmetry of the modified effective Lagrangian $L_{inv} = L_{eff} + L_{WZ}$.

On the other hand, another approach, apparently unrelated to Wess–Zumino idea, was introduced in order to make classical, *massive*, Proca theory gauge invariant, and is known as the Stückelberg compensating formalism [4].

We conjecture that both above-mentioned methods could be *equivalent*. Our motivation is that both methods achieve the same goal, i.e. to restore a broken gauge invariance of an *Abelian* vector theory.

In this letter we would like to clarify the conjectured equivalence between Stückelberg and Wess–Zumino methods. Let us start by recalling the Wess–Zumino idea of restoring gauge invariance in anomalous effective gauge theories. To maintain pedagogical transparency we shall start from the simple and well known 2D effective anomalous Lagrangian of the Schwinger model [5] which is given by

$$L_{eff} = -\frac{e^2}{4\pi} \left[F_{\mu\nu} \frac{1}{\square} F^{\mu\nu} - 2\mathbf{a} A^\mu A_\mu \right] \quad (1)$$

(1) was obtained from a Dirac fermion matter Lagrangian, having both vector and axial $U(1)$, *classical*, gauge symmetries. In order to treat the quantum breaking of classical symmetries on the same footing we have introduced an arbitrary, regularization dependent, parameter \mathbf{a} [6] in the effective Lagrangian (1). The effective Lagrangian L_{eff} varies, under gauge transformation $\delta_\Lambda A_\mu = \partial_\mu \Lambda$, as

$$\delta_\Lambda L_{eff} = -\frac{e^2}{\pi} \mathbf{a} \Lambda(x) \partial_\mu A^\mu \neq 0 \quad (2)$$

Eq.(2) shows the gauge non-invariance of the 2D effective Lagrangian, which is the origin of the gauge anomaly. One could object that the gauge invariance can be restored by a suitable choice of the arbitrary parameter \mathbf{a} as $\mathbf{a} = 0$. However, this choice would simply shift the anomaly from the vector to the axial current. We want to show how *both* axial and vector symmetries can be restored simultaneously at the quantum level, for any choice of \mathbf{a} . Therefore, we shall keep the parameter \mathbf{a} arbitrary throughout the paper.

In order to restore gauge invariance in (1), Wess and Zumino have proposed to construct a suitable, local, term whose gauge variation cancel the gauge variation of the effective Lagrangian. A straightforward guess for the Wess–Zumino Lagrangian would be

$$L_{WZ} \equiv \mathbf{a} \frac{e^2}{\pi} \phi(x) \partial_\mu A^\mu \quad (3)$$

where, one introduces the scalar field $\phi(x)$ transforming as $\delta_\Lambda \phi(x) = \Lambda(x)$ under gauge transformations. However, careful examination of such Wess–Zumino Lagrangian shows that

it is gauge non-invariant, not only due to the non-invariance of ϕ , but also due to the explicit dependence on the gauge field A . Therefore, additional kinetic term for ϕ field is needed to compensate non-invariance coming from the gauge field itself, leading to the correct form of the 2D Wess–Zumino Lagrangian for the Schwinger model:

$$L \equiv -\frac{e^2}{4\pi} [-4\mathbf{a}\phi(x)\partial_\mu A^\mu + 2\mathbf{a}\phi\Box\phi] \quad (4)$$

Once the modified effective Lagrangian L_{inv} is defined as

$$L_{inv} \equiv L_{eff} + L_{WZ} = -\frac{e^2}{4\pi} \left[F_{\mu\nu} \frac{1}{\Box} F^{\mu\nu} - 2\mathbf{a} A^\mu A_\mu - 4\mathbf{a}\phi(x)\partial_\mu A^\mu + 2\mathbf{a}\phi\Box\phi \right] \quad (5)$$

then, one can verify that the addition of Wess–Zumino piece ensures the gauge invariance of the total effective Lagrangian for *any choice* of the parameter \mathbf{a} . In this way, the Wess–Zumino idea achieves its goal. On the other hand, a quick look at (5) shows that it can be re-written as

$$L_{inv} = -\frac{e^2}{4\pi} \left[F_{\mu\nu} \frac{1}{\Box} F^{\mu\nu} - 2\mathbf{a} (A^\mu - \partial^\mu \phi)(A_\mu - \partial_\mu \phi) \right] \quad (6)$$

Thus, starting from gauge non-invariant Lagrangian (1), and making it gauge invariant following Wess–Zumino prescription, we got the invariant effective Lagrangian (6), which is nothing else but the gauge invariant massive Lagrangian originally proposed by Stückelberg for Proca theory. Reversing the conclusion, we have shown that the advocated Stückelberg origin of the Wess–Zumino terms leads to the *equivalence* of the two methods.

Encouraged by this toy model, one can proceed further and generalize the above result to *any* Abelian anomalous gauge theory. Notice that the relation between Wess–Zumino terms in the effective Lagrangian and Stückelberg mass term is specific to 2D since it is only in 2D that a quantum anomaly induces a mass term for the gauge field. Anomalies in higher dimensions do not give origin to the mass term, but a simple generalization of the Stückelberg idea allows to proceed along the same line as in 2D to prove the equivalence of the Wess–Zumino and Stückelberg approach. In fact, Stückelberg compensation of non-invariant, Abelian, vector theories is achieved by the substitution of the complete field A_μ with the combination $A_\mu - \partial_\mu \phi$, where ϕ is a compensating scalar field transforming as $\delta_\Lambda \phi = \Lambda$. from the previous discussion it follows that to implement the compensation mechanism one needs the *explicit* form of the anomalous effective Lagrangian. Recently, we have obtained the general form of the anomalous Lagrangian in arbitrary even dimensional ($D \equiv 2k$) space-time:

$$L_{eff} = \sum_{m=0}^{m_{max}} \frac{g^{2m+1} e^{k-2m}}{(2\pi)^k} \epsilon^{\mu_1 \dots \mu_{2k-4m} \nu_1 \dots \nu_{4m}} F_{\mu_3 \mu_4} \dots F_{\mu_{2k-4m-1} \mu_{2k-4m}} \\ \times F_{\nu_1 \nu_2}^5 \times \dots \times F_{\nu_{4m-1} \nu_{4m}}^5 \left[F_{\mu_1 \mu_2} \frac{1}{\Box} \left(\partial^\mu A_\mu^5 \right) - 2\mathbf{a} A_{\mu_1} A_{\mu_2}^5 \right] \quad (7)$$

where, e and g are the vector and axial coupling constants. Except than in the very special 2D case, both the vector field A_μ and the axial field A_μ^5 are independent gauge potentials, corresponding to anomalous local symmetries. m_{max} is a maximal integer number

compatible with the restriction $0 \leq m \leq (k-1)/2$, imposed by the anomalous Feynman diagrams in even dimensions. Details can be found in Ref. [7]. As a check, one can obtain the Schwinger model anomalous effective Lagrangian (1) by putting $k = 1$, $m = 0$ in (7) and exploiting the $2D$ relation $A_\mu = \epsilon_{\mu\nu} A^{\nu 5}$.

Now we are ready to address the Stückelberg derivation of Wess–Zumino terms in full generality. A gauge invariant Stückelberg Lagrangian can be obtained from (7) by replacing A_μ and A_μ^5 with the corresponding compensated potentials:

$$L_{inv}^{Stuck.} = \sum_{m=0}^{m_{max}} \frac{g^{2m+1} e^{k-2m}}{(2\pi)^k} \epsilon^{\mu_1 \dots \mu_{2k-4m} \nu_1 \dots \nu_{4m}} F_{\mu_3 \mu_4} \dots F_{\mu_{2k-4m-1} \mu_{2k-4m}} \\ \times F_{\nu_1 \nu_2}^5 \times \dots \times F_{\nu_{4m-1} \nu_{4m}}^5 \left[F_{\mu_1 \mu_2} \frac{1}{\square} \partial^\mu (A_\mu^5 - \partial_\mu \phi^5) - 2 \mathbf{a} (A_{\mu_1} - \partial_{\mu_1} \phi) (A_{\mu_2}^5 - \partial_{\mu_2} \phi^5) \right] \quad (8)$$

Dealing with two independent vector and axial vector gauge fields, one needs *two, independent* Stückelberg compensators ϕ and ϕ^5 . Straightforward manipulations of the above Lagrangian lead to the equivalent Wess–Zumino form:

$$L_{inv} = L_{eff} + L_{WZ} \\ = L_{eff} + \sum_{m=0}^{m_{max}} \frac{g^{2m+1} e^{k-2m}}{(2\pi)^k} \epsilon^{\mu_1 \dots \mu_{2k-4m} \nu_1 \dots \nu_{4m}} F_{\mu_3 \mu_4} \dots F_{\mu_{2k-4m-1} \mu_{2k-4m}} \\ \times F_{\nu_1 \nu_2}^5 \times \dots \times F_{\nu_{4m-1} \nu_{4m}}^5 \left[(\mathbf{a} - 1) F_{\mu_1 \mu_2} \phi^5 - \mathbf{a} F_{\mu_1 \mu_2}^5 \phi \right] \\ \equiv L_{eff} - \left[X^{5\rho} \partial_\rho \phi^5 + X^\rho \partial_\rho \phi \right] \quad (9)$$

where, we neglected surface terms. In the Wess–Zumino part of the Lagrangian we have introduced the following definitions ¹

$$X^{5\rho} \equiv 2(\mathbf{a} - 1) \sum_{m=0}^{m_{max}} \frac{g^{2m+1} e^{k-2m}}{(2\pi)^k} \epsilon^{\rho \mu_2 \dots \mu_{2k-4m} \nu_1 \dots \nu_{4m}} F_{\mu_3 \mu_4} \dots F_{\mu_{2k-4m-1} \mu_{2k-4m}} \\ \times F_{\nu_3 \nu_4}^5 \dots F_{\nu_{4m-1} \nu_{4m}}^5 \left[A_{\mu_2} F_{\nu_1 \nu_2}^5 + \frac{2m}{k} (A_{\mu_2}^5 F_{\nu_1 \nu_2} - A_{\mu_2} F_{\nu_1 \nu_2}^5) \right] \quad (10)$$

and

$$X^\rho \equiv -2\mathbf{a} \sum_{m=0}^{m_{max}} \frac{g^{2m+1} e^{k-2m}}{(2\pi)^k} \epsilon^{\mu_1 \dots \mu_{2k-4m} \nu_1 \dots \nu_{4m}} F_{\mu_5 \mu_6} \dots F_{\mu_{2k-4m-1} \mu_{2k-4m}} F_{\nu_3 \nu_4}^5 \dots F_{\nu_{4m-1} \nu_{4m}}^5 \\ \times \left[\delta_{\mu_3}^\rho F_{\nu_1 \nu_2}^5 A_{\mu_4} F_{\mu_1 \mu_2}^5 + \frac{2m+1}{k} (\delta_{\mu_1}^\rho A_{\mu_2}^5 F_{\mu_3 \mu_4} F_{\nu_1 \nu_2}^5 - \delta_{\nu_1}^\rho A_{\nu_2} F_{\mu_3 \mu_4}^5 F_{\mu_1 \mu_2}^5) \right] \quad (11)$$

One indeed recognizes the additional terms in (9) as Wess–Zumino terms. This gives the proof of equivalence of the two methods for arbitrary spacetime dimension. For the record,

¹ Eq.(11) and eq.(10) are defined as the most general expressions which allow to transform $\epsilon \phi F$ into $X \partial \phi$ eq.(9). Dependence on the parameter m counts the number of ways in which this can be done.

only the term $X^{5\mu}$ with $\mathbf{a} = 0$ can be found in the literature. One is indeed free to preserve at the quantum level one of the two classical symmetries by a proper choice of the regularization method. Accordingly, the anomaly is conventionally shifted in the divergence of the axial current. However, we have proven that one can restore *both* symmetries simultaneously, for any value of \mathbf{a} , through the construction of independent Wess–Zumino terms. This result may be of particular importance for chiral models where *none* of the classical symmetries can be preserved at the quantum level [8].

In a recent paper we have studied a Proca theory where gauge symmetry is explicitly broken by a *classical*, mass term. This theory has been made gauge invariant by the introduction of the Stückelberg scalar compensator, and proven to be dual to the $B \wedge F$ model [9]. On the other hand, in the first part of the present paper we have extended the notion of Stückelberg compensation to gauge non-invariant anomalous theories at the quantum level, and proven the equivalence between the Stückelberg and Wess–Zumino methods. Combining the results of the present paper and the ones in [9], we conjecture that the duality ideas of the latter should apply to the former. We mean that the scalar compensator in the Wess–Zumino terms, in (9), can be dualized to a Kalb–Ramond-like $p = 2k - 2$ forms. In order to achieve this we adopt general dualization procedure described in [9]. One starts from a suitably chosen parent Lagrangian L_P which, in the present case, turns out to be of the form ²

$$\begin{aligned}
L_P = & -\frac{1}{2(2k-1)!} H_{\mu_1 \dots \mu_{2k-1}} H^{\mu_1 \dots \mu_{2k-1}} + \frac{M}{(2k-1)!} \epsilon^{\mu_1 \dots \mu_{2k}} A_{\mu_{2k}} H_{\mu_1 \dots \mu_{2k-1}} \\
& - \frac{1}{(2k-1)!} \left(M H^{\mu_1 \dots \mu_{2k-1}} + \frac{1}{f} X^{\mu_1 \dots \mu_{2k-1}} \right) F^*_{\mu_1 \dots \mu_{2k-1}}(\phi) \\
& - \frac{1}{2(2k-1)!} H^5_{\mu_1 \dots \mu_{2k-1}} H^5{}^{\mu_1 \dots \mu_{2k-1}} + \frac{M_5}{(2k-1)!} \epsilon^{\mu_1 \dots \mu_{2k}} A^5_{\mu_{2k}} H^5_{\mu_1 \dots \mu_{2k-1}} \\
& - \frac{1}{(2k-1)!} \left(M_5 H^5{}^{\mu_1 \dots \mu_{2k-1}} + \frac{1}{f_5} X^{5\mu_1 \dots \mu_{2k-1}} \right) F^*_{\mu_1 \dots \mu_{2k-1}}(\phi^5) \quad (12)
\end{aligned}$$

where, $X^{\mu_1 \dots \mu_{2k-1}}$ and $X^{5\mu_1 \dots \mu_{2k-1}}$ are Hodge duals of (11) and (10). ϕ and ϕ^5 are Stückelberg scalar compensators which will be dualized to Kalb–Ramond-like fields. H and H^5 are a priori *independent* fields in the parent Lagrangian (12). We have also introduced dimensionless constants f, f_5 (numbers) in front of X and X^5 in order to identify *quantum* symmetry breaking terms at various stages of calculation. These factors are not present in the original Lagrangian (9). Thus, in order to achieve invariance of the anomalous theory we have to take $f = f_5 = 1$ when calculating appropriate variations.

It is worth mentioning the important point regarding the construction of L_P . The parent Lagrangian must have certain symmetry properties which will be reflected in the (dual) theories derived from it. Let us require the gauge invariance of the complete Lagrangian L_{inv} as $\delta_\Lambda L_{inv} \equiv \delta_\Lambda L_P + \delta_\Lambda L_{eff} = 0$. Then, it turns out that the variations of the terms involving

²We assign canonical dimensions (in units of mass) as follows: $[A] = k - 1$, $[\phi] = k - 2$.

This choice implies dimensional coupling constants $[e] = [g] = 2 - k$, and Wess–Zumino terms $[\partial X] = k + 2$. Accordingly, kinetic terms have dimension $2k$. Moreover, in the parent Lagrangian we require $[H] = k$.

classical mass parameters cancel among themselves in (12). On the other hand, variations of the (quantum) terms involving dimensionless f and f_5 cancel against the variations of the effective Lagrangian (7).

The dualization proceeds as follows: varying (12) with respect to H we find the solution

$$H^{\mu_1 \dots \mu_{2k-1}} = M \left(\epsilon^{\mu_1 \dots \mu_{2k-1} \mu_{2k}} A_{\mu_{2k}} - F_{\mu_1 \dots \mu_{2k-1}}^*(\phi) \right) \quad (13)$$

$$H^5{}_{\mu_1 \dots \mu_{2k-1}} = M_5 \left(\epsilon^{\mu_1 \dots \mu_{2k-1} \mu_{2k}} A_{\mu_{2k}}^5 - F_{\mu_1 \dots \mu_{2k-1}}^*(\phi^5) \right) \quad (14)$$

and re-inserting the solution back into L_P , we find the Stückelberg-like model

$$\begin{aligned} L_{VA} = & -\frac{M^2}{2} (A_\mu - \partial_\mu \phi)^2 - \frac{M_5^2}{2} (A_\mu^5 - \partial_\mu \phi^5)^2 \\ & - \frac{1}{f(2k-1)!} F_{\mu_1 \dots \mu_{2k-1}}^*(\phi) X^{\mu_1 \dots \mu_{2k-1}} - \frac{1}{f_5(2k-1)!} F_{\mu_1 \dots \mu_{2k-1}}^*(\phi^5) X^5{}_{\mu_1 \dots \mu_{2k-1}} \end{aligned} \quad (15)$$

The first two terms in the Lagrangian (15) are gauge invariant mass terms, while the last two terms correspond to Wess–Zumino terms of (9). In this way both classical and quantum non-invariances have been improved by the same scalar compensator ϕ transforming as $\delta_\Lambda \phi = \Lambda$.

The Lagrangian dual to (15) is obtained by varying the parent Lagrangian (12) with respect to both scalars ϕ, ϕ^5 . In this way we get

$$H_{\mu_1 \dots \mu_{2k-1}} = \frac{1}{M} \left(\partial_{[\mu_1} B_{\mu_2 \dots \mu_{2k-1}]} - \frac{1}{f} X_{\mu_1 \dots \mu_{2k-1}} \right) \quad (16)$$

$$H^5{}_{\mu_1 \dots \mu_{2k-1}} = \frac{1}{M_5} \left(\partial_{[\mu_1} B_{\mu_2 \dots \mu_{2k-1]}^5 - \frac{1}{f_5} X_{\mu_1 \dots \mu_{2k-1}}^5 \right) \quad (17)$$

Re-inserting 16 and 17 back into (12) we find the dual Lagrangian:

$$\begin{aligned} L_{dual} = & -\frac{1}{2(2k-1)! M^2} \left(\partial_{[\mu_1} B_{\mu_2 \dots \mu_{2k-1}]} - \frac{1}{f} X_{\mu_1 \dots \mu_{2k-1}} \right)^2 \\ & - \frac{1}{(2k-1)!} \epsilon^{\mu_1 \dots \mu_{2k}} \left(\partial_{[\mu_1} B_{\mu_2 \dots \mu_{2k-1}]} - \frac{1}{f} X_{\mu_1 \dots \mu_{2k-1}} \right) A_{\mu_{2k}} \\ & - \frac{1}{2(2k-1)! M_5^2} \left(\partial_{[\mu_1} B_{\mu_2 \dots \mu_{2k-1]}^5 - \frac{1}{f_5} X_{\mu_1 \dots \mu_{2k-1}}^5 \right)^2 + \\ & - \frac{1}{(2k-1)!} \epsilon^{\mu_1 \dots \mu_{2k}} \left(\partial_{[\mu_1} B_{\mu_2 \dots \mu_{2k-1]}^5 - \frac{1}{f_5} X_{\mu_1 \dots \mu_{2k-1}}^5 \right) A_{\mu_{2k}}^5 \end{aligned} \quad (18)$$

which can be suitably re-written as

$$L_{dual} = L_{BF} + L_{WZ} \quad (19)$$

$$\begin{aligned} L_{BF} = & -\frac{1}{2(2k-1)! M^2} \left[H_{\mu_1 \dots \mu_{2k-1}}(B) \right]^2 - \frac{1}{(2k-1)!} \epsilon^{\mu_1 \dots \mu_{2k}} H_{\mu_1 \dots \mu_{2k-1}}(B) A_{\mu_{2k}} \\ & - \frac{1}{2(2k-1)! M_5^2} \left[H_{\mu_1 \dots \mu_{2k-1}}^5(B) \right]^2 - \frac{1}{(2k-1)!} \epsilon^{\mu_1 \dots \mu_{2k}} H_{\mu_1 \dots \mu_{2k-1}}^5(B) A_{\mu_{2k}}^5 \end{aligned} \quad (20)$$

$$\begin{aligned}
L_{WZ} = & -\frac{\left(X_{\mu_1 \dots \mu_{2k-1}}\right)^2}{2M^2 f^2 (2k-1)!} + \frac{X^{\mu_1 \dots \mu_{2k-1}}}{f (2k-1)!} \left(\epsilon_{\mu_1 \dots \mu_{2k-1} \mu_{2k}} A^{\mu_{2k}} + \frac{1}{M^2} H_{\mu_1 \dots \mu_{2k-1}}(B) \right) \\
& -\frac{\left(X_{\mu_1 \dots \mu_{2k-1}}^5\right)^2}{2M_5^2 f_5^2 (2k-1)!} + \frac{X^5{}^{\mu_1 \dots \mu_{2k-1}}}{f_5 (2k-1)!} \left(\epsilon_{\mu_1 \dots \mu_{2k-1} \mu_{2k}} A^5{}^{\mu_{2k}} + \frac{1}{M_5^2} H_{\mu_1 \dots \mu_{2k-1}}^5(B) \right)
\end{aligned} \tag{21}$$

L_{dual} has been split in two terms: L_{BF} corresponds to the classical $B \wedge F$ (without quantum anomaly), which is dual to the mass term in (15); L_{WZ} provides the dual version of the Wess–Zumino terms in (15). Thus, (18) and (15) are dual to each other, and we can summarize the effects of the dualization as follows:

Classical Stückelberg model Let us start without Wess–Zumino terms, or quantum anomaly in the effective Lagrangian. Only classical mass terms M and M_5 break gauge symmetries. They are made invariant by the scalar compensator ϕ a la’ Stückelberg in (15). Their dual form is given by (20) in terms of the $B \wedge F$ model, in agreement with the results in [9].

Quantum Stückelberg model Once classical mass terms are made gauge invariant, we switch on the quantum anomaly. In (15) the quantum non-invariance of the effective Lagrangian is improved by the Wess–Zumino terms expressed through ϕ and ϕ^5 . In the dual version of the theory (21) the scalar compensators ϕ and ϕ_5 are replaced by the rank $p = 2k - 2$ Kalb–Ramond field B . One can see from (18) that the Kalb–Ramond-like field *is* in fact a Stückelberg compensator of $X(A)$ in the dual version of the theory. As a compensator the B field must vary under vector and axial vector gauge transformation³. On general grounds, if $\delta H(B) = f^{-1} \delta X$ and $\delta X = \partial(\Lambda G(A, A^5))$, then, variation of Kalb–Ramond-like field is $\delta B = f^{-1} \Lambda G(A, A^5)$. The explicit form of $G(A, A^5)$ can be extracted from the transformations of B under vector and axial vector symmetry. From (11) and (10) one finds

$$\begin{aligned}
\delta_\Lambda B_{\alpha_3 \dots \mu_{2k}} = & -\frac{2\mathbf{a}}{f} \Lambda \sum_{m=0}^{m_{max}} \frac{g^{2m+1} e^{k-2m}}{(2\pi)^k (2k-2)!} \delta_{[\alpha_3 \dots \alpha_{2k-4m} \dots \alpha_{2k}] }^{[\mu_3 \dots \mu_{2k-4m} \nu_1 \dots \nu_{4m}]} \left(1 - \frac{2m+1}{k} \right) \times \\
& \times F_{\mu_3 \mu_4}^5 F_{\mu_5 \mu_6} \dots F_{\mu_{2k-4m-1} \mu_{2k-4m}} F_{\nu_1 \nu_2}^5 \dots F_{\nu_{4m-1} \nu_{4m}}
\end{aligned} \tag{22}$$

$$\begin{aligned}
\delta_{\Lambda_5} B_{\alpha_3 \dots \mu_{2k}} = & -\frac{2\mathbf{a}}{f} \Lambda_5 \sum_{m=0}^{m_{max}} \frac{g^{2m+1} e^{k-2m}}{(2\pi)^k (2k-2)!} \delta_{[\alpha_3 \dots \alpha_{2k-4m} \dots \alpha_{2k}] }^{[\mu_3 \dots \mu_{2k-4m} \nu_1 \dots \nu_{4m}]} \left(\frac{2m+1}{k} \right) \\
& \times F_{\mu_3 \mu_4} F_{\mu_5 \mu_6} \dots F_{\mu_{2k-4m-1} \mu_{2k-4m}} F_{\nu_1 \nu_2}^5 \dots F_{\nu_{4m-1} \nu_{4m}}
\end{aligned} \tag{23}$$

³ There is also an additional tensor gauge transformation of the $p = 2k - 2$ Kalb–Ramond fields described as

$$\begin{aligned}
\delta B_{\mu_1 \dots \mu_{2k-2}} &= \partial_{[\mu_1} \Omega_{\mu_2 \dots \mu_{2k-2}]} \\
\delta B_{\mu_1 \dots \mu_{2k-2}}^5 &= \partial_{[\mu_1} \Omega_{\mu_2 \dots \mu_{2k-2}}^5]
\end{aligned}$$

which is still an invariance of the dual Lagrangian, and it is *anomaly free*. Furthermore, in case of classical duality there is no need for additional gauge transformations of B since $B \wedge F$ term is automatically gauge invariant.

$$\delta_{\Lambda} B_{\alpha_3 \dots \mu_{2k}}^5 = \frac{2(\mathbf{a} - 1)}{f_5} \Lambda \sum_{m=0}^{m_{max}} \frac{g^{2m+1} e^{k-2m}}{(2\pi)^k (2k-2)!} \delta_{[\alpha_3 \dots \alpha_{2k-4m} \dots \alpha_{2k}]}^{[\mu_3 \dots \mu_{2k-4m} \nu_1 \dots \nu_{4m}]} \left(1 - \frac{2m}{k}\right) \times F_{\mu_3 \mu_4} F_{\mu_5 \mu_6} \dots F_{\mu_{2k-4m-1} \mu_{2k-4m}} F_{\nu_1 \nu_2}^5 \dots F_{\nu_{4m-1} \nu_{4m}}^5 \quad (24)$$

$$\delta_{\Lambda_5} B_{\alpha_3 \dots \mu_{2k}}^5 = \frac{2(\mathbf{a} - 1)}{f_5} \Lambda_5 \sum_{m=0}^{m_{max}} \frac{g^{2m+1} e^{k-2m}}{(2\pi)^k (2k-2)!} \delta_{[\alpha_3 \dots \alpha_{2k-4m} \dots \alpha_{2k}]}^{[\mu_3 \dots \mu_{2k-4m} \nu_1 \dots \nu_{4m}]} \left(\frac{2m}{k}\right) \times F_{\mu_1 \mu_2} F_{\mu_5 \mu_6} \dots F_{\mu_{2k-4m-1} \mu_{2k-4m}} F_{\nu_3 \nu_4}^5 \dots F_{\nu_{4m-1} \nu_{4m}}^5 \quad (25)$$

The above transformations guarantee the invariance of the complete Lagrangian $\delta L_{inv} \equiv \delta L_{eff} + \delta L_{dual} = 0$, putting $f = f_5 = 1$. Equations (18) and (15) show the dualization “flipping” $M \rightarrow 1/M$ and $M_5 \rightarrow 1/M_5$, which forbids the massless limit in the dual version of the theory. In our model classical masses act as coupling constants in (12), and are reversed by the duality transformation.

For the sake of transparency, let us look at the four dimensional case ($k = 2, m = 0$) where (22), (23), (24) and (25) give

$$\delta_{\Lambda} B_{\mu\nu} = -\frac{\mathbf{a}}{f} \frac{ge^2 \Lambda}{(2\pi)^2} F_{\mu\nu}^5 \quad (26)$$

$$\delta_{\Lambda_5} B_{\mu\nu} = \frac{\mathbf{a}}{f} \frac{ge^2 \Lambda_5}{(2\pi)^2} F_{\mu\nu} \quad (27)$$

$$\delta_{\Lambda} B_{\mu\nu}^5 = \frac{\mathbf{a} - 1}{f_5} \frac{ge^2 \Lambda}{(2\pi)^2} F_{\mu\nu} \quad (28)$$

$$\delta_{\Lambda_5} B_{\mu\nu}^5 = 0 \quad (29)$$

For the choice $\mathbf{a} = 0$ the above formulae reproduce results of [10]. One can verify that the gauge transformations of the Kalb–Ramond compensator in (29) agree with the ones found in supergravity [11]. This result suggests both a natural SUSY extension of the gauge theory discussed in this paper, and gives further support to the role of the Kalb–Ramond fields as a Stückelberg compensator, i.e. field transforming under vector and axial vector gauge transformations.

In summary, we have shown the *equivalence* of the Stückelberg and Wess–Zumino methods to restore the gauge invariance of an *anomalous* Abelian theory of massive one-forms. Combining this equivalence with the results about classical duality between Stückelberg and $B \wedge F$ theories, we have found a new, dual, form of Wess–Zumino terms. Massless limits $M \rightarrow 0$, $M_5 \rightarrow 0$ cannot be performed as a consequence of the flipping between “strong/weak coupling” regimes. Thus, in order to produce dual theory one has to have classical mass terms form the start.

The application of the method described in this letter to non–Abelian, anomalous gauge theories is currently under investigation. We are developing a non–Abelian dualization procedure acting on Yang–Mills fields coupled to Kalb–Ramond tensors [12]. What we are looking for is the generalization of the anomalous effective action (7) in the non–Abelian case. Once this goal will be brought to a success, we shall be able to build up the non–Abelian version of the model discussed in this letter.

REFERENCES

- [1] S.E.Treiman, R. Jackiw, B.Zumino, E.Witten “*Current Algebra and Anomlies*”, World Scientific, (1985)
- [2] T.Berger “*Fermions in two (1+1)-dimensional anomalous gauge theories: the chiral Schwinger model and the chiral quantum gravity*”, Hamburg U. DESY-90-084, July 1990.
- [3] J.Wess, B.Zumino Phys. Lett. **37B**, 95, (1971)
L.D. Faddeev Phys. Lett. **145B**, 81, (1984)
L.D. Faddeev, S.L. Shatashvili Phys. Lett. **167B**, 225, (1986)
O.Babelon, F.Schaposnik, C.Viallet Phys. Lett. **177B**, 385, (1986)
K.Harada, I. Tsutsui Phys. Lett. **183B**, 311, (1987)
- [4] E.C.G. Stückelberg, Helv. Phys. Acta **11**, 299, (1938)
- [5] S.E.Treiman, R. Jackiw, D.J.Gross *Lectures on Current Algebra and its Applications*, Princeton UP, Princeton NJ, (1972)
- [6] A.Smailagic, R.E.Gamboa-Saravi Phys.Lett. **B192**, 145, (1987)
- [7] A.Smailagic, E.Spallucci “*Higher Dimensional Schwinger-like Anomalous Effective Lagrangian*”, hep-th/0003293, to appear in Phys.Rev.**D**
- [8] R. Jackiw, R.Rajaraman Phys.Rev.Lett. **54**, 1219, (1985)
- [9] A.Smailagic, E.Spallucci Phys.Rev. **D61**, 067701, (2000)
- [10] S.Deguchi, T.Mukai, T.Nakajima Phys.Rev. **D59**, 065003, (1999)
- [11] E.Bergshoeff, M.de Roo, B.de Wit, P.van Nieuwenhuizen Nucl.Phys. **B195** 97 (1982)
- [12] A.Smailagic, E.Spallucci “*Dualization of non-Abelian $B \wedge F$ model*”, Univ. TS preprint (2000)